Labelled Model Modal Logic

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1 Introduction

There is no agreement among researchers in the area of automated deduction about which features (besides computational efficiency) a suitable theorem proving system for non-classical (in particular modal) logics should have. In our opinion, such a system should (A) avoid *ad hoc* manipulation of the modal formulas; (B) provide a simple and uniform treatment of a wide variety of modal (and other non-classical) logics; (C) form an adequate basis for developing efficient proof search methods; (D) yield proofs according to familiar, natural inference patterns; (E) provide an explicit model construction. In this paper we describe a modal theorem proving system, that we call KEM, which appears to satisfy all this requirements: it treats the full modal language; it works for a wide class of normal modal logics (and it can be extended to other non-classical logics, e.g. temporal or conditional logics); and it forms a basis for combining both efficiency and naturalness. Moreover it automatically generates models using a label formalism to bookkeep "world" paths. KEM results from the combination of two kinds of rules: rules for processing the propositional part (which are the same for all modal logics), and rules for manipulating labels according to the accessibility relations for the given logics. The key features of KEM are outlined as follows.

2 Label Formalism

A set $\Im$ of "world" labels is introduced, where a label i is defined to be either (i) an element of a (non empty) set $\Phi_C = \{w_1, w_2, w_3, \ldots\}$ of constant "world" symbols, or (ii) an element of a (non empty) set $\Phi_V = \{W_1, W_2, W_3, \ldots\}$ of "world" variables, or (iii) a "path" term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (m', m)$ where (m', m) is a label. Intuitively, we may think of a label $i \in \Phi_C$as denoting a world, and a label $i \in \Phi_V$ as denoting a set of worlds in some Kripke model (we assume familiarity with standard Kripke semantics, see [Ch80]). A label $i = (k', k)$ may be viewed as representing a path from k to a (set of) world(s) k' accessible from k. For example, the label (W_1, w_1) represents a path which takes us to a set W_1 of worlds accessible from the initial world w_1; $(w_2, (W_1, w_1))$ represents a path which takes us to a world w_2 accessible by any world accessible from w_1, (i.e., accessible by the subpath (W_1, w_1)) and so on (notice that the labels are read from right to left). For any label $i = (k', k)$ we call k' the head of i, k the body of i, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive: if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. For example, if i is $(w_4, (W_3, (w_3, (W_2, w_1))))$, then $b(i) = (W_3, (w_3, (W_2, w_1)))$, $b(b(i)) = (w_3, (W_2, w_1))$, $b(b(b(i))) = (W_2, w_1)$, $b(b(b(b(i)))) = w_1$. We call each of $b(i)$, $b(b(i))$, etc., a segment of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$. For any label i, we define the length of i, $l(i)$, as the number of world symbols in i. We call a label i restricted if $h(i) \in \Phi_C$, otherwise we call it unrestricted.

3 Basic Unifications

We define a substitution in the usual way as a function $\sigma : \Phi_V \to \Im^-$ where $\Im^- = \Im - \Phi_V$. For two labels i, k and a substitution σ we shall use $(i, k)\sigma$ to denote both that i and k are unifiable (briefly, are σ-unifiable) and the result of their unification. On this basis we define several specialised, logic-dependent notions of σ-unification.

$(i, k)\sigma^K = (i, k)\sigma \iff$

(i) at least one of i and k is restricted, and

(ii) for every $s(i)$, $s(k)$, $l(s(i)) = l(s(k))$, $(s(i), s(k))\sigma^K$

$(i,k)\sigma^D = (i,k)\sigma$

$(i,k)\sigma^T = (s(i),k)\sigma \iff$

$\quad l(i) > l(k)$, and $\forall h(s(i)) : l(s(i)) \geq l(s(k)),\ (h(s(i)),h(k))\sigma = (h(i),h(k))\sigma$ or

$(i,k)\sigma^T = (i,s(k))\sigma \iff$

$\quad l(k) > l(i)$, and $\forall h(s(k)) : l(s(k)) \geq l(s(i)),\ (h(i),h(s(k)))\sigma = (h(i),h(k))\sigma$

$(i,k)\sigma^{X4} = h(k) \times h(b(k)) \times (\ldots \times (t^*(k) \times (i,s(k))\sigma^X)\ldots)) \iff$

$\quad l(i) \leq l(k)$ and $(i,s(k))\sigma^X,\ h(i) \in \Phi_V$, or

$(i,k)\sigma^{X4} = h(i) \times h(b(i)) \times (\ldots \times (t^*(i) \times (s(i),sk)\sigma^X)\ldots)) \iff$

$\quad l(k) \leq l(i)$ and $(s(i),k)\sigma^X,\ h(k) \in \Phi_V$

where $t^*(k)$ (resp. $t^*(i)$) denotes the element of k (resp. i) which immediately follows $s(k)$ (resp. $s(i)$) and $X = K, D$.

$$(i,k)\sigma^{S4} = \begin{cases} (i,k)\sigma^T & h(shortest\{i,k\} \in \Phi_C \\ (i,k)\sigma^{D4} & h(shortest\{i,k\} \in \Phi_V \end{cases}$$

$(i,k)\sigma^{X5} = (h(i),h(k))\sigma^X \times (b(b(i)),b(k))\sigma^L \iff$

$\quad (h(i),h(k))\sigma^X$ and $(b(b(i),b(k))\sigma^L$ if $h(i) \in \Phi_V$, or

$(i,k)\sigma^{X5} = (h(i),h(k))\sigma^X \times (b(i),b(b(k)))\sigma^L \iff$

$\quad (h(i),h(k))\sigma^X$ and $(b(i),b(b(k)))\sigma^L$ if $h(k) \in \Phi_V$

where

$$\sigma^L = \begin{cases} \sigma^X \text{ or } \sigma^{X5} & \text{if } l(i) = l(k) \\ \sigma^{X5} & \text{if } l(i) \neq l(k) \end{cases}$$

for $X = K, D$ and, if $X = K$, at least one of $h(i), h(k), h(b(i)), h(b(k))$ is in Φ_C.

$(i,k)\sigma^{S5} = (h(i),h(k))\sigma$

For $L = 4, 5, B$ we define the L-reduction of a label i to be a function $r_L : \Im \to \Im$ determined as follows:

$$r_4(i) = \begin{cases} (h(i), b(b(i))) & i \text{ restricted} \\ (h(i), r_4(b(i))) & \text{otherwise} \end{cases}$$

$$r_B(i) = \begin{cases} b(b(i)), & i \text{ unrestricted and } l(i) > 2 \\ (h(i), r_B(b(i))), & i \text{ restricted} \end{cases}$$

$$r_5(i) = \begin{cases} (h(i), b(b(i))) & \text{if } i, b(i) \text{ unrestricted} \\ (h(i), r_5 b(i)) & \text{otherwise} \end{cases}$$

We are now able to define the notion of two labels $i,\ k$ being σ_L-unifiable

(i) $L = K, D, S5$

$\quad (i,k)\sigma_L \Leftrightarrow (i,k)\sigma^L$

(ii) $L = K4, KB, K5, K45, K5B, D4, DB, D5, D45, T, B, S4$

$\quad (i,k)\sigma_L \Leftrightarrow$ either

$\quad\quad$ (i) $(i,k)\sigma^*$, or

$\quad\quad$ (ii) $(i, r_L(k))\sigma^*$, or

$\quad\quad$ (iii) $(r_L(i), k)\sigma^*$, or

$\quad\quad$ (iv) $(r_L(i), r_L(k))\sigma^*$

where

for $l(i) = l(k)$	for $l(i) \neq l(k)$
$\sigma^* = \sigma^K, L = K4, KB, K5, K45, K5B$	$\sigma^* = \sigma^T, L = T, B$
$\sigma^* = \sigma^D, L = D4, DB, D5, D45, T, B, S4$	$\sigma^* = \sigma^{X4}, L = K4, D4, S4.$
$\sigma^* = \sigma^{X5}, L = K5, D5$	$\sigma^* = \sigma^{X5}, L = K5, D5, K45, D45, K5B.$

The following table gives a complete picture of the logics we are considering.

K	D	T	$K4$	$D4$	$S4$	$K5$	$D5$	$S5$	KB	DB	B	$K45$	$D45$	$K4B$
			r_4	r_4	r_4	r_5	r_5		r_B	r_B	r_B	r_4	r_4	r_4, r_B
σ^K	σ^D	σ^T	σ^{K4}	σ^{D4}	σ^{S4}	σ^{K5}	σ^{D5}	σ^{S5}	σ^K	σ^D	σ^T	σ^{K5}	σ^{D5}	σ^{K5}

The notions of L-reduction and σ^L-unification are meant to mirror the formal properties of the accessibility relation in the Kripke semantics for the various modal logics. For example the notions of σ^K- and σ^D-unification are related in an obvious way to the idealization condition. Thus, $(w_2, (W_1, w_1))$, $(W_3, (W_2, w_1))$ are σ^D-unifiable but not σ^K-unifiable (since the segments $(W_1, w_1), (W_2, w_1)$ are not σ^K-unifiable by condition (i) of the above definition), while they are. The reason is that in the "non idealisable" logic K the "denotations" of W_1 and W_2 may be empty (i.e., there can be no worlds accessible from w_1), which obviously makes their unification impossible, while in the "idealisable" logic D they are not empty, which makes them to be unifiable "on" any constant. For the notion of σ^T-unification take for example $i = (w_3, (W_2, (w_2, w_1)))$ and $k = (w_3, (W_1, w_1))))$. Here $(W_2, w_3)\sigma = (w_3, w_3)\sigma$. Then i and k σ^T-unify to $(w_3, (w_2, w_1))$. This intuitively means that the world accessible from a subpath $s(i) = (W_2, (w_2, w_1))$ after deletion of the "irrelevant" (because of reflexivity) step to an arbitrary world in the set W_2 are accessible from any path k which turns out to be identical with $s(i)$. For the notion of σ^{X4}-unification take for example $i = (W_3, (w_2, w_1))$ and $k = (w_5, (w_4, (w_3, (W_2, w_1))))$. Here $s(k) = (w_3, (W_2, w_1))$. Then i and k σ^{K4}-unify to $(w_5, (w_4, (w_3, (w_2, w_1))))$ since $((W_3, (w_2, w_1)), (w_3, (W_2, w_1)))\sigma^K$. This intuitively means that all the worlds accessible from a subpath $s(k)$ of k are accessible from any path i which turns out to be identical with $s(k)$.

4 Rules of KEM

The following formulation uses Smullyan-Fitting's "α, β, ν, π" unifying notation for signed formulas. A signed formula followed by an arbitrary label will be called a *labelled signed formula (LS-formula).* As usual X^C is used to denote the conjugate of a signed formula X, i.e. the result of changing the sign of X to its opposite. Two LS-formulas $X, i,\ Z, k$ such that $Z = X^C$ and $(i, k)\sigma_L$ will be called σ_L-complementary.

0-premise rule: $$\frac{}{X \qquad X^C}\ PB\ [i \text{ restricted}]$$

1-premise rules: $$\frac{\alpha, i}{\alpha_n, i}\ (n = 1, 2) \qquad \frac{\nu, i}{\nu_0, (i', i)}\ [\, i' \in \Phi_V, i' \text{ new}] \qquad \frac{\pi, i}{\pi_0(i', i)}\ [\, i' \in \Phi_C, i' \text{ new}]$$

2-premise rules: $$\frac{\begin{array}{c}\beta, i\\ \beta_1^C, k\end{array}}{\beta_2, (i, k)\sigma_L}\ [(i, k)\sigma_L] \qquad \frac{\begin{array}{c}\beta, i\\ \beta_2^C, k\end{array}}{\beta_1, (i, k)\sigma_L}\ [(i, k)\sigma_L] \qquad \frac{\begin{array}{c}X, i\\ X^C, k\end{array}}{\times\ (i, k)\sigma_L}\ [(i, k)\sigma_L]$$

Here the α-rules are just the usual linear branch-expansion rules of the tableau method, while the β-rules correspond to such common natural inference patterns as *modus ponens, modus tollens*, etc. The rules for the modal operators bear a not unexpected resemblance to the familiar quantifier rules of the tableau method. "i' new" in the proviso for the ν- and π-rule obviously means: i' must not have occurred in any label yet used. Notice that in all inferences via an α-rule the label of the premise carries over unchanged to the conclusion, and in all inferences via a β-rule the labels of the premises must be σ_L-unifiable, so that the conclusion inherits their unification. PB (the "Principle of Bivalence") represents the (LS-version of the) semantic counterpart of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any given world). PNC (the "Principle of Non-Contradiction") corresponds to the familiar branch-closure rule of the tableau method, saying that from a contradiction of the form (occurrence of a pair of σ_L-complementary LS-formulas) $X, i,\ X^C, k$ on a branch we may infer the closure of the branch. The $(i, k)\sigma_L$ in the "conclusion" of

PNC means that the contradiction holds "in the same world".

5 Proof search

The following definitions are extensions to the modal case of those given for the classical case, [Mo88]. By a KEM-tree we mean a tree generated by the rules of KEM. Given a branch τ of a KEM-tree we shall call an LS-formula X, i *E-analysed in a branch* τ if either (i) X is of type α and both α_1, i and α_2, i occur in τ; or (ii) X is of type β and one of the following conditions is satisfied: (a) if β_1^C, k occurs in τ and $(i,k)\sigma_L$, then also $\beta_2, (i,k)\sigma_L$ occurs in τ, (b) if β_2^C, k occurs in τ and $(i,k)\sigma_L$, then also $\beta_1, (i,k)\sigma_L$ occurs in τ; or (iii) X is of type ν and $\nu_0, (i', i)$ occurs in τ for some $i' \in \Phi_V$ not previously occurring in τ, or (iv) X is of type π and $\pi_0, (i', i)$ occurs in τ for some $i' \in \Phi_C$ not previously occurring in τ. We shall call a branch τ of a KEM-tree *E-completed* if every LS-formula in it is E-analysed and there are no complementary formulas which are notσ_L-complementary. Finally, we shall call an LS-formula X, i of type β *fulfilled in a branch* τ if either β_1, i' or β_2, i' occur in τ, where either (i) $i' = i$, or (ii) i' is obtained from i by instantiating $h(i)$ to a constant not occurring in i, or (ii i) $i' = (i,k)\sigma_L$ for some $\beta_n^C, k,\ n = 1, 2$, such that $(i,k)\sigma_L$. We shall say that a branch τ of a KEM-tree is *completed* if it is both E-completed and all the LS-formulas of type β in it are fulfilled, it is *σ_L-closed* if it ends with an application of PNC. We shall call a KEM-tree *completed* if every branch is completedand it is σ_L-closed if all its branches are σ_L-closed. A *L-proof* of a formula A is a σ_L-closed KEM-tree starting with FA, i. Since KEM's basic control structure is non deterministic, to build a KEM proof search algorithm we have to rely on the procedure for the *canonical* KEM-trees. A KEM-tree is canonical iff all the applications of 1-premise rules come before the applications of 2-premise rules, which preceed the applications of the 0-premise rule.

The following procedure starts from the 1-branch, 1-node tree consisting of FA, i and applies the rules of KEM until the resulting KEM-tree is either closed or completed. At each stage of proof search (i) we choose an open non completed branch τ. If τ is not E-completed, then (ii) we apply the 1-premise rules until τ becomes E-completed. If the resulting branch τ' is neither closed nor completed, then (iii) we apply the 2-premise rules until τ becomes E-completed. If the resulting branch τ' is neither closed nor completed, then (iv) we choose an LS-formula of type β which is not yet fulfilled in the branch and apply PB so that the resulting LS-formulas are β_1, i' and β_1^C, i' (or, equivalently β_2, i' and β_2^C, i'), where $i = i'$ if i is restricted, otherwise i' is obtained from i by instantiating $h(i)$ to a constant not occurring in i; (v) if the branch is not E-completed nor closed because there are complementary formulas which are notσ_L-complementary, then we apply PB to one of the two complementary formulas with a restricted label which occurs previously in the branch, and which unifies with one of the labels of the complementary formulas, (vi) we repeat the procedure in each branch generated by PB.

Remark 1 Notice that in this procedure PB is applied *only* to immediate signed subformulas of LS-formulas which occur (unfulfilled) in the chosen branch, and *only* when the branch has been E-completed. Such a *restricted* use of the cut rule removes from the search space the redundancy generated by the standard tableau branching rules. Indeed it is easy to see that the given procedure makes all choices in such a way that at each step of proof search the search space is as small as possible, while preserving the subformula property of proofs.

Remark 2 It is worth to note that each tree is a (class of) Hintikka's model(s) where the labels denote worlds (i.e., Hintikka's modal sets), and the operations (unifications and reductions) on labels behave according to the conditions placed on the accessibility relations for L.

It can be shown (see [ACG94]) the following

THEOREM 1 A KEM-tree for a formula A of L is closed iff the canonical KEM-tree for A is closed.

THEOREM 2 A canonical KEM-trees always terminates.

This theorem follows from the fact that the subformulas of a formula A are finite in number and the number of labels which can occur in the KEM-tree for A is limited by the number of modal operators in A.

6 Soundness and Completeness

We shall show that the KEM versions of the logics L we have been considering are equivalent to their respective axiomatic formulations. In order to do this, we have to prove (i) that the characteristic axioms and the inference rules of the axiomatic L are derivable in KEM, and (ii) that the rules of KEM are derived rules in the axiomatic L. To prove (ii) we show that the rules of KEM hold in a model for the respective L.

Let $\mathcal{F} =< \mathcal{G}, R >$ be a Kripke frame and let $\mathcal{M} =< \mathcal{G}, R, v >$ be a Kripke model with the usual conditions on their elements; R is defined as $\Gamma R \Gamma' \Leftrightarrow \{A : \Box A \in \Gamma\} \subseteq \Gamma'$, where Γ denotes an element of the not empty set $\mathcal{G}$; and v is as usual.

We now define a translation function g from labels to the model's frame as follows:

$g : \Im \to \mathcal{F}$ so that:

(a) If $i \in \Phi_C$ then $g(i) = \exists \Gamma \in \mathcal{G}$

b) If $i \in \Phi_V$ then $g(i) = \forall \Gamma \in \mathcal{G}$

(c) If $l(i) = n$ then we shall denote by i^m the $h(j)$ such that $l(j) = m,\ m \leq n$, and j is a segment of i; whence

$g(i) = Qg(i^1)Qg(i^2)(g(i^1)Rg(i^2) \wedge Qg(i^3)(g(i^2)Rg(i^3) \wedge \cdots \wedge Qg(i^n)(g(i^{n-1}Rg(i^n))))$

where Q denotes either $\forall$ or $\exists$ according to what kind of label is its i^m, $\overline{Q}$ will denote $\exists(\forall)$ if Q is $\forall(\exists)$, and R is the frame's relation.

Let f be the translation function from LS-formulas to the model defined as:

$f(SA, i) = g(i), v(A, g(h(i))) = S$

LEMMA 1. For any $i \in \Im$ and $L = 4, B, 5$ if X, i then $X, r_L(i)$.

Proof $L = 4$. We analyse only the relevant cases. Let us suppose that $X, i \Rightarrow X, r_4(i)$ doesn't hold; then X, i will be true and $X, r_4(i)$ will be false. If i is restricted also $r_4(i)$ is restricted and $h(r_4(i)) = h(i)$, thus obtaining a contradiction. If i is unrestricted we put $g(h(i)) = \Gamma$, $g(h(b(i))) = \Gamma'$ and $g(h(b(b(i)))) = \Gamma''$; from the definition of f and the hypothesis it follows that $\ldots Q\Gamma''(\ldots \wedge Q\Gamma'(\Gamma'' R \Gamma' \wedge \forall\Gamma(\Gamma' R \Gamma)))$, so that $v(A, \Gamma) = S$ and $\overline{Q}\Gamma''(\ldots \wedge \exists\Gamma(\Gamma' R \Gamma)), v(A, \Gamma) = S^C$. From the former we obtain $v(A, \Gamma) = S \Leftrightarrow v(\Box A, \Gamma') = S$ but $\Box A \to \Box\Box A$ holds in each world, therefore $v(\Box\Box A, \Gamma') = S \Leftrightarrow v(\Box A, \Gamma) = S$ and, because of transitivity, $v(\Box A, \Gamma'') = S$. From the latter we obtain $v(A, \Gamma) = S^C \Leftrightarrow v(\Box A, \Gamma'') = S^C$ which contradicts the result following from the truth of X, i.

$L = B$. We only prove the case(s) in which i is unrestricted. Let us suppose that $X, i \Rightarrow X, r_B(i)$ doesn't hold, whence, by putting $g(h(i)) = \Gamma$, $g(h(b(i))) = \Gamma'$ and $g(h(b(b(i)))) = \Gamma''$, we obtain $\ldots \forall\Gamma(\Gamma' R \Gamma), v(A, \Gamma) = X$ and $g(i) = \ldots Q\Gamma''(\ldots \wedge Q\Gamma'((\Gamma'' R \Gamma') \wedge \forall\Gamma(\Gamma' R \Gamma)))$; but, by the symmetry of the model, this implies also $\Gamma' R \Gamma''$ and hence $v(A, \Gamma'') = S$, from which, since our hypothesis states $v(A, \Gamma'') = S^C$, we get a contradiction.

$L = 5$. Let us suppose that $X, i \Rightarrow X, r_5(i)$ doesn't hold, whence, by putting $g(h(i)) = \Gamma$, $g(h(b(i))) = \Gamma'$ and $g(h(b(b(i)))) = \Gamma''$, we obtain $\ldots Q\Gamma''(\ldots \forall\Gamma'(\Gamma'' R \Gamma') \wedge \forall\Gamma(\Gamma' R \Gamma)), v(A, \Gamma) = S$ and $\ldots Q\Gamma'' \forall\Gamma(\Gamma'' R \Gamma), v(A, \Gamma) = S^C$; but, by the euclideannes of the model and the predicate calculus we get a contradiction.

LEMMA 2. For any $i, k \in \Im$ and $L = K, K4, K5, D5, T, B, S4, S5$, if X, i and $(i, k)\sigma^L$, then X, k.

Proof. The proof is by induction on the length of the labels. If $\min\{l(i), l(k)\} = 1$, then at least one of i and k is either a constant or a variable, so that five cases will be present, by the definition of unifications according to our label expansion rules: i, k are either i) two constants, or ii) a variable and a constant, or iii) two variables, or iv) a variable and a label, or v) a constant and a label.[1]

Case i) For $L = K, D, S5$, let us suppose that X, i, X^C, k and $(i, k)\sigma^L$, but two constants unify if and only if they are the same constant, and so $i = k$; therefore from the hypothesis and the definition of v and f we get $v(A, g(i)) = S$ and $v(A, g(k)) = S^C$, and also $g(i) = g(k)$, thus obtaining a contradiction.

Case ii) If $i(k)$ is a variable and $k(i)$ is a constant, then there exists a substitution σ so that $i\sigma = k(k\sigma = i)$, whence $(i, k)\sigma^L$, from which the result follows by the same argument as case i).

[1] Cases ii), iii), and iv) are not found in KEM proofs, but they are useful both for dealing with cases in the inductive step and for case v).

Case iii) and iv) These cases are identical to the previous one because: 1) each variable unifies with any label, and 2) $\mathcal{G}$ is not empty.

Case v) For $L = T, S4, S5$. Let us assume, for the sake of economy, that $l(i) = 1$ and $l(k) > 1$; for $S5$ we have $(i,k)\sigma^{S5}$; this mean $(h(i), h(k))\sigma^{S5}$ but $h(i) = i$, therefore $h(k) \in \Phi_V$ and so this case falls under case ii) due to the equivalence relation of the model. For $L = T, S4$ if $(i,k)\sigma^L$ then each $h(s(k))$ so that $l(s(k)) > 1$ belongs to Φ_V, therefore we have $g(k) = \forall g(h(s^2(k)))((g(i)Rg(h(s^2(k))) \wedge \ldots$-$\wedge \forall g(h(k))(g(h(b(k)))Rg(h(k))))$ then through reflexivity and the definition of v and f repeating the argument of case i) we get the result we want.

For the inductive step we have $\min\{l(i), l(k)\} = n > 1$. Let us assume inductively that the lemma is valid up to n; if $l(i) = l(k)$ we shall write i and k as $(h(i), b(i))$ and $(h(k), b(k))$, respectively. Given that $(i,k)\sigma^L$, we get by the inductive hypothesis that $(b(i), b(k))\sigma^L$; thus we have to analyse only $h(i)$ and $h(k)$ which, if $L = K$, fall under the cases i) and ii); for $L = D, S5$ we have to examine case iii). Since two variables unify on any label *i.e.*, $\forall i \in \Im, (h(i))\sigma = j$ and $(h(i))\sigma = j$, we get $g(h((b(i), b(k))\sigma^L)) = \exists\Gamma$, and thus, by the seriality condition which holds in all the standard models of the logics we are considering, we obtain $\exists\Gamma : \Gamma R\Gamma'$, so that $g(h(i))$ and $g(h(k))$ are not empty, like their intersection. If we suppose that the lemma doesn't hold we get by $f, v(A, g(h(i))) = S$ and $v(A, g(h(k))) = S^C$, but then there exists a world belonging to $g(h(i)) \cap g(h(k))$ where a formula A is true and false at the same time.

$L = K4, D4, S4$. If $l(i) < l(k)$ (the case in which $l(k) < l(i)$ is managed in the same way) we shall write k as $(h(k), (\ldots, s(k))\ldots)$ where $l(i) = l(s(k))$. By hypotheses $(i, k\sigma^L$ whence $(i, s(k))\sigma^L$ and $h(i) \in \Phi_V$; moreover, let us suppose that the lemma doesn't hold. By the definitions of f and g, if we put $g(h((i, s(k))\sigma^L)) = \Gamma$, then $f(X, i) = \ldots\forall\Gamma(g(h(b(i)))R\Gamma), v(A, \Gamma) = S$ and $f(X^C, k) = \ldots Q\Gamma(g(h(b(i)))R\Gamma \wedge \ldots \wedge g(h(b(k)))Rg(h(k)), v(A, g(h(k))) = S^C$ from which, by transitivity and the predicate calculus, we obtain $\ldots Qg(h(k))(g(h((i, s(k))\sigma^L))Rg(h(k)), v(A, g(h(k))) = S^C$, which leads to a contradiction. For $S4$ we have to deal with another case which is the case a) of T bellow.

$L = T$. Let us suppose that the lemma does not hold. If $l(i) < l(k)$ (the case $l(k) < l(i)$ is analogous) we shall write i and k respectively as $(h(i), b(i))$ and $(h(k), (\ldots, s(k))\ldots)$ where $l(b(i)) = l(s(k))$, we shall analyse two cases: a) $h(i) \in \Phi_C$; b) $h(i) \in \Phi_V$. a) Due to hypotheses $(i,k)\sigma^T$ whence $(b(i), s(k))\sigma$ and $(h(i), t^*(k))\sigma = (h(i), h(k))\sigma$ but this is the same as case v) of the inductive base. b) By hypotheses $(i,k)\sigma^L$ whence $(b(i), s(k))\sigma$ and for each $s^*(k)$ so that $l(s^*(k)) > l(b(i))(h(i), h(s^*(k)))\sigma = (h(i), h(k))\sigma$ but this is possible if and only if at least one of the $h(s^*(k)) \in \Phi_C$; by $v, f$and g we get $f(X, i) = \ldots\forall g(h(i))(g(h((b(i), (s(k))\sigma))Rg(h(i))), v(A, g(h(i))) = S$ and if we denote $g(h((b(i), (s(k))\sigma))$ by Γ then we have $f(X^C, k) = \ldots\forall g(h(i))(\Gamma Rg(h(i))) \wedge \ldots \wedge \exists g(h(s^*(k))(\ldots g(h(b(k)))Rg(h(k))), v(A, h(k)) = S^C$ from which, by reflexivity and the predicate calculus, we obtain $\ldots\forall g(h(i))\exists g(h(s(k)))(\Gamma Rg(h(s^*(k))), v(A, g(h(s^*(k)))) = S^C$, which leads to a contradiction.

$L = X5$. Let us suppose that the lemma does not hold. From hypotheses we have $(i,k)\sigma^{X5}$ and then $(h(i), h(k))\sigma$ and either $(b(i), b(b(k)))\sigma^L$ or $(b(k), b(b(i)))\sigma^L$; moreover, we know that $h(i) \neq h(k)$ (if they are equal then $i = k$). Therefore we have two cases a) the two head are two variable; b) they are one variable and one constant. By definitions of v, g, and f, and if we put either $(b(i), b(b(k)))\sigma^L$ or $(b(k), b(b(i)))\sigma^L$ with Γ, we obtain $f(X, i) = \ldots Qg(h(i))(\Gamma Rg(h(i))), v(A, h(i)) = S$ and $f(X^C, k) = \ldots Q\Gamma(g(h(b(k)))((\Gamma Rg(h(b(k)))) \wedge Qg(h(k))(g(h(b(k)))Rg(h(k)))), v(A, h(i)) = S^C$; due to the euclideannes of the model $Qg(h(i))(g(h(b(i)))Rg(h(i)))$, this implies that there exists a world belonging to $g(h(i)) \cap g(h(k))$ in which a formula A is at the same time true and false. $g(h(i)) \cap g(h(k))$ is not empty by the facts that at least one among $h(i), h(k)$ is a variable and $(i,k)\sigma^{X5}$. If either $(b(i), b(b(k)))\sigma^{X5}$ or $(b(k), b(b(i)))\sigma^{X5}$ we can repeat the above reasoning as far as we arrive to deal with $(s(i), s(k))\sigma$ for which the inductive hypothesis holds.

THEOREM 3. $\vdash_L A \Leftrightarrow \vdash_{KEM(L)} A$ for $L = K, K4, KB, K45, K4B, K5, D, D4, DB, D45, D5$,-$T, B, S4, S5$. *Proof*$\Rightarrow$ The characteristic axioms of L and *modus ponens* are provable in KEM (see [AG93] and [DM91] for a proof that *modus ponens* is a derived rule in KE, the propositional subsytem of KEM). We give a KEM-proof of the rule of necessitation. Let us assume that $\vdash_{KEM(L)} A$. Then the following is KEM-proof of $\Box A$.

1. $F\Box A, w_1$

2. $FA, (w_2, w_1)$

3. $\times(w_2, w_1)$

Proof$\Leftarrow$ The α-rules and PB are obviously derived rules in L. For theβ-rules and PNC. By the hypothesis: $(i,k)\sigma_L$ and hence, by the above lemmas and the definitions of the σ_L-unifications, the formulas involved have the same value in i (k) and $(i,k)\sigma_L$; after that these rules become rules of KE, and thus they are derived rules in L. For ν-rules: let us suppose $\nu = T\Box A$; if we put $g(h(i)) = \Gamma$ and $g(h(i', i)) = \Gamma'$, then $v(\Box A, \Gamma) = T$; but $v(\Box A, \Gamma) = T \Leftrightarrow \forall\Gamma' : \Gamma R\Gamma', v(A, \Gamma') = T$, and $(\forall\Gamma' : \Gamma R\Gamma',$-$v(A, \Gamma') = T) = f(\nu_0, (i', i))$ with i' unrestricted. The proof of $E\pi$ is similar.

7 Final remarks

Our approach exploits various ideas already occurring in the literature ([Cat91], [DG93], [Wri85]). A sequent based modal proof system using *indexed formulas* has been proposed by Jackson and Reichgelt [JR89]. This system is the most closely related to ours. The index formalism is almost identical, but the unification algorithm used to resolve complementary formulas in the various modal logics does not work for the non-idealisable K logics. Similar ideas are found in Wallen's matrix-connection method [Wal90]. This is probably the most refined automated proof search system for non-classical logics currently available. However it yields proofs in a natural inference-style (e.g. in the form of sequent or tableau proofs) only derivatively and works only for a few standard modal logics. In conclusion, we believe that KEM has several advantages over most current modal theorem proving systems. Its PROLOG implementation (see [ACG94]) shows that the label unification scheme it uses is natural and efficient. Moreover several extensions of KEM to handle multi modal logics with various interaction between modalities are under work.

8 References

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