Section 1: Proof Theory and Categorical Logic

A Reduplication and Loop Checking Free Proof System for $S4$

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Most of the sequent/tableau based proof systems for the modal logic $S4$ need to reduplicate formulas and thus are required to adopt some method of loop checking. In what follows we present a tableau–like proof system for $S4$, based on D'Agostino and Mondadori's classical KE, which is free of reduplication and loop checking. The key feature of this system (let us call it $KES4$) consists in its use of (i) a label formalism which models the semantics of the modal operators according to the usual conditions for $S4$; and (ii) a label unification scheme which tells us when two labels "denote" the same world in the $S4$–model(s) generated in the course of proof search. Moreover, it uses special closure conditions to check models for putative contradictions. $KES4$'s label scheme uses two kinds of atomic labels, respectively constant (i.e. elements of $\Phi_C = \{w_1, w_2 \ldots\}$) and variable (i.e. elements of $\Phi_C = \{w_1, w_2 \ldots\}$) "world" symbols. En element i of the set $\Im$ of "world" labels is defined to be either (i) an element of Φ_C, or (ii) an element of Φ_V, or (iii) a "path" term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (m', m)$ where (m', m) is a label. $KES4$'s label unification scheme involves two kinds of unifications, respectively "high" (σ^L) and "low" (σ_L) unifications. In general, "high" unifications are meant to mirror specific accessibility constraints and they are used to build "low" unifications, which account for the full range of conditions governing the appropriate accessibility relation. As far as $S4$ is concerned these are defined as follows:

$$(i,k)\sigma^T = (i, s(k))\sigma \iff$$
$$l(k) > l(i), \text{ and } \forall h(s(k)) : l(s(k)) \geq l(s(i)), (h(i), h(s(k)))\sigma = (h(i), h(k))\sigma$$

$$(i,k)\sigma^{D4} = h(k) \times h(b(k)) \times (\ldots \times (t^*(k) \times (i, s(k))\sigma^X)\ldots)) \iff$$
$$l(i) \leq l(k) \text{ and } (i, s(k))\sigma^X,\ h(i) \in \Phi_V, \text{ or}$$

$$(i,k)\sigma^{D4} = h(i) \times h(b(i)) \times (\ldots \times (t^*(i) \times (s(i), sk)\sigma^X)\ldots)) \iff$$
$$l(k) \leq l(i) \text{ and } (s(i), k)\sigma^X,\ h(k) \in \Phi_V$$

$$(i,k)\sigma^{S4} = \begin{cases} (i,k)\sigma^T & h(shortest\{i,k\}) \in \Phi_C \\ (i,k)\sigma^{D4} & h(shortest\{i,k\}) \in \Phi_V \end{cases}$$

where $t^*(k)$ (resp. $t^*(i)$) denotes the element of k (resp. i) which immediately follows $s(k)$ (resp. $s(i)$). We define the $S4$-reduction of a label i to be a function $r_4 : \Im \rightarrow \Im$ determined as follows:

$$r_4(i) = \begin{cases} (h(i), b(b(i))) \\ i,\ h(i) \in \Phi_V, l(i) \leq 3 \end{cases}$$

The notion of two labels $i,\ k$ being σ_{S4}-unifiable is now defined as

$$(i,k)\sigma_{S4} = \begin{cases} (r_4(i,k))\sigma & l(i)=l(k) \\ (r_4(i,k))\sigma^{S4} & l(i)\neq l(k) \end{cases}$$

The rules of $KES4$ combine the usual tableau and natural deduction elimination rules with rules for manipulating labels. They are formulated as follows:

$$\frac{\alpha,k}{\alpha_i,k}[i=1,2] \quad \frac{\beta,k \quad \beta_i^C,l}{\beta_{3-i},m}[m=(k,l)\sigma_{S4}] \quad \frac{\nu,k}{\nu_0,(k',k)} \quad \frac{\pi,k}{\pi_0,(k'',k)}$$

where $k'\in\Phi_V, k''\in\Phi_C$ and they have not been previously used in the proof. In addition to the above operational rule we have the following two 'structural' rules called respectively PNC and PB

$$\frac{X,k \quad X^C,l}{\times,m}[m=(k,l)\sigma_{S4}] \quad \frac{}{X,k \qquad X^C,k}[h(k)\in\Phi_C]$$

As we have proved elsewhere, the above set of rules is sound and complete for $S4$. Moreover a proof search algorithm can be given for canonical $KES4$-trees. A $KES4$-tree is said to be *canonical* if it is generated by applying the inference rules in the following fixed order: first the α-, ν-, π-rules, then the β-rules and PNC, and finally PB. The application of PB is allowed only on subformulas of the β formulas which occur unanalysed in the branch, where a β formula is said to be *analysed* in a branch τ if either (i) if β_1^C,l occurs in τ and $(k,l)\sigma_{S4}$, then also $\beta_2,(k,l)\sigma_{S4}$ occurs in τ, or (ii) if β_2^C,l occurs in τ and $(k,l)\sigma_{S4}$, then also $\beta_1,(k,l)\sigma_{S4}$ occurs in τ. The proof search for the modal case differs from that for the classical case in that the condition of "fulfilment" for β formulas (i.e. that either β_1 or β_2 occurs in the branch) is dropped. Let us define a *block* in a $KES4$-tree as (roughly) the set of formulas resulting from applying the α-, ν-, π-rules as fas as possible. It can be seen that the restriction (implicit in the definition of a canonical $KES4$-tree) which constrains a β rule to be used at most once in its block together with the fact that no formula is analysed more than once, prevents both reduplications and loops. By way of an example we try to $KES4$-refute the Löb axiom.

1. $F\Box(\Box A\to A)\to\Box A$	w_1
2. $T\Box(\Box A\to A)$	w_1
3. $F\Box A$	w_1
4. $T\Box A\to A$	(W_1,w_1)
5. FA	(w_2,w_1)
6. $F\Box A$	(w_2,w_1)
7. FA	$(w_3,(w_2,w_1))$

Notice that if the restriction did not hold we could have applied the β-rule to (4) and (7) thus obtaining $F\Box A,(w_3,(w_2,w_1))$ which implies, by an application of the ν–rule, $FA,(w_4,(w_3,(w_2,w_1)))$ and therefore we must have applied the β-rule again thus obtaining $F\Box A,(w_4,(w_3,(w_2,w_1)))$ and so on.