Labelled Modal Sequents

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1 Introduction

Gabbay (1996) proposes a new methodology called *Labelled Deductive Systems* (*LDS*) to deal, uniformly, with logical systems. This approach, where formulas are indexed with labels to bring metalevel features in the object language, is very flexible: it enables us to work not only on the logical part, but also on the labels (using an appropriate algebra), and both.

Gentzen systems are often used to define calculi as well as consequence relations. Nevertheless, such systems do not work so well when intensional operators are involved. In order to generalise them to modal logic, the most direct course is to try and devise rules for $\Box$ of the same kind as those governing the classical operators; in other words to force the classical pattern on the modal operator. Moreover, there is no single interpretation of modality: each of them requires its own consequence relation. This leads to the fact that modal sequents are far to be uniform (see Goré (1995) for an overview of such systems). Labelled sequents seem to offer an higher degree of uniformity, at least for classes of logics. Unfortunately almost all recent works proposing labels in sequent systems suffer from the same illness: they use labels properties (semantics) to reduce modal consequence to classical one. So they fail to provide a general system for defining real notions of modal deducibility.

In the spirit of *LDS* we develop a general framework for modal sequent calculi ($\mathscr{LMS}$) that provides a notion of modal consequence. This is achieved as follows:

- There is only a rule for modality, and the rules for the boolean connectives are generalized to the modal case.
- All the modal inferences are kept in the labels, no external constraints of modal axioms are needed.

We use KEM label language (see Artosi et al. (1998); Governatori (1997)) that simulates accessibility relation, and an algorithm to determine the conditions under which two labels can be compared. If so, we can apply inference rules on the related formulas. We have two kinds of atomic labels *constants* — $w_1, w_2, \ldots$ — and *variables* — $W_1, W_2, \ldots$ — that might be combined into *path* labels. Roughly a constant corresponds to $\Diamond$ and a variable to $\Box$. A path is a label with the following form (i, i'), where i is an atomic label and i' is either a path or a constant, in the same way an atomic label corresponds to a single modality a path corresponds to

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a string of modalities. It is worth noting that labels may be split so that the parts can be considered separately. Another interesting feature of the present approach is that boolean and modal combinations of labelled formulas are permitted; so, if A, B are well formed formulas, and i, j, k are labels, then $A : i$, $B : j$ and $(A : i \rightarrow B : j) : k$ are expressions of our language. This fact forces us to reconsider the classical rules for propositional connectives. For example we have the following instance of *modal modus tollens*: $(A : i \rightarrow B : j) : k$ and $\neg\Box B : l$ imply $\neg A : m$, under the appropriate conditions on the labels. In particular l and k should be comparable and j corresponds to $\Box$.[1]

2 Labelled Modal Language

As we have already alluded to, we allow boolean and modal combination of labelled formulas, so we first introduce the appropriate label formalism and then we extend the language of modal logic to the labelled case.

2.1 Label Formalism

KEM has two basic kinds of atomic labels: variables and constants. The label scheme arises from such a basic alphabet, so that a "world" label is either a world-symbol or a "structured" sequence of world-symbols that we call a "world-path". Constant and variable world-symbols denote worlds and sets of worlds respectively (in a Kripke model), while a world-path conveys information about access between the worlds in it. KEM labels are built in a modular way and so they can be easily composed and decomposed. Furthermore, we shall use auxiliary "dummy" labels, that allow world-paths to be split into proper sub-paths.

Definition 1. Let $\Phi_A = \{w_0, w_0', \dots\}$ be a not empty set of auxiliary or actual world symbols; let $\Phi_C = \{w_1, w_2, \dots\}$ be a not empty set of constant world symbols (or simply constants); let $\Phi_V = \{W_1, W_2, \dots\}$ be a not empty set of variable world symbols (or simply variables). The set $\Im$ of label is then defined as follows

$$\begin{aligned} \Im \;\; &= \;\; \bigcup_{1 \le i} \Im_i \text{ where } \Im_i : \\ & \Im_1 = \Phi_A; \\ & \Im_{n+1} = (\Phi_C \cup \Phi_V) \times \Im_n, \;\; (n > 1). \end{aligned}$$

According to the above definition a label is either (i) an element of the set Φ_A, or (ii) a path term (k', k) where (iia) $k' \in \Phi_C \cup \Phi_V$ and (iib) $k \in \Phi_C$ or $k = (i', i)$ where (i', i) is a label. From now on we shall use $i, j, k, \dots$ to denote arbitrary labels.

For any label $i = (k', k)$ we shall call k' the *head* of i, k the *body* of i, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive (they correspond to projection functions): if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. We call each of $b(i)$, $b(b(i))$, etc., a *segment* of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$. We shall call a label i *restricted* if $h(i) \in \Phi_C$, otherwise *unrestricted.*

[1]See section 4.2 for the actual definition of the modal modus tollens.

For any label i, we define the length of i, $\ell(i)$, as the number of world-symbols in i, i.e., $\ell(i) = n \Leftrightarrow i \in \Im_n$. $s^n(i)$ will denote the segment of i of length n, i.e., $s^n(i) = s(i)$ such that $\ell(s(i)) = n$. We shall use $h^n(i)$ as an abbreviation for $h(s^n(i))$.

For any label $i, \ell(i) > n$, we define the *countersegment-n* of i, as follows:

$$c^n(i) = h(i) \times (\cdots \times (h^k(i) \times (\cdots \times (h^{n+1}(i), j))))(n < k < l(i))$$

where j is an auxiliary label. In other words the countersegment-n of a label i is the label obtained from i by replacing $s^n(i)$ with an auxiliary world symbol.

There is a strict relationship between labels and possible world semantics. The intuitive reading of a constant is a possible world in a Kripke models, while a variable denotes a set of worlds. A path label moreover encodes the information about the accessibility relation. Indeed a label such as $(W_1, (w_1, w_0))$ represents the set of worlds accessible from the world denoted by w_1, which itself is accessible from the actual world w_0. An auxiliary world symbol stands for an actual world.

In general a label corresponds to the model generated from a formula with respect to the actual world: the actual world of the label. However, sometimes, we want to change our point of view, so we move our actual world inside a path, and to consider the truncated model. This effect is achieved by the notions of segment and countersegment. We split a label into two parts: the segment is the path which leads us to the current actual world from the previous one; the countersegment then is the truncated model.

2.2 Labelled Well-formed Formulas

The standard modal language L is extended by attaching to each well-formed formula of L (wff) a KEM label. So, the notion of *label formula* is defined as follows:

DEFINITION 2.

- if A is a wff and i is a label, then $A : i$ is a *labelled formula* (lwff for short);
- if $A : i$ is a lwff and j is a label, then $A : i : j$ is a lwff;
- if $A : i$ and $B : j$ are lwff's, $\#$ is a binary connective, and k is a label, then $(A : i \# B : j) : k$ is a lwff.
- if $A : i$ is a lwff and j is a label, then $\Box(A : i) : j$, $\Diamond(A : i) : j$ and $\neg(A : i) : j$ are lwff's.

Formulas without labels will be considered labelled with the auxiliary label w_0; so A will be regarded as $A : w_0$.

According to Smullyan-Fitting (Fitting, 1983) unifying notation that classifies formulas we shall say that

DEFINITION 3. Two wffs A, and B of type ν and π are *complementary* iff A_0 and B_0 are *complementary.*

In the previous section we have seen that the labels can be decomposed. Here we show how labels can be composed. Given a lwff $A : i : j$ we can compose i and j in a label k which satisfies the following conditions:

$$i = c^{\ell(j)}(k) \qquad\qquad j = s^{\ell(j)}(k) \tag{1}$$

2.3 From Labels to Modalities

In this section we shall examine the relationships between labels and modalities.

Our rules are designed in such a way that each modal step depends on the properties of the labels involved, which are defined to simulate the syntactical structure of modal formulas.

Why should we use then labels instead modalities? The algebra of labels is extremely flexible and allows easy manipulations of them. However, sometimes, it may be useful to deal also with modalities mixed with labels, at least we want to translate the final steps of proofs in a plain modal language. Another example where we use mixed expressions is the generalization of classical principles such as *modus ponens*, *modus tollens* to the modal case, where a part of the inference pattern is expressed in label notation and the other uses modalities. To this end we need a function which translates labels into modalities.

We shall use $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \ldots$ for strings of positive modalities; let $\mathfrak{M}$ be the set of positive modalities, by definition of modality the empty string of positive modalities is a modality, we use $\sharp$ to denote it.

Definition 4. Let ϕ^+ be a map from $\Im$ to $\mathfrak{M}$ thus defined:

$$\phi^+(i) = \begin{cases} \sharp & i \in \Im_1 \\ \Box\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_V \\ \Diamond\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_C \end{cases} \tag{2}$$

Let ϕ^- be a map from $\Im$ to $\mathfrak{M}$ thus defined:

$$\phi^-(i) = \begin{cases} \sharp & i \in \Im_1 \\ \Diamond\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_V \\ \Box\phi(b(i)) & \ell(i) > 1 \text{ and } h(i) \in \Phi_C \end{cases} \tag{3}$$

3 Unifications

The key feature of our approach is that in the course of proof labels are manipulated in a way closely related to the semantics of modal operators and "matched" using a specialized unification algorithm. That two labels i and k are unifiable means, intuitively, that the set of worlds they "denote" have a non-null intersection. The basic element of the unification is the substitution function which maps each variable in label to a label, and each constant to itself. Formally

$$\begin{aligned} \sigma \quad : \quad & \Phi_V \mapsto \Phi_A \cup \Phi_C \cup \Phi_V \\ & \mathbf{1}_{\Phi_C \cup \Phi_A} \end{aligned}$$

Applying the substitution recursively in a label we obtain the substitution of a label

$$\sigma(i) = \begin{cases} \sigma(i) & \ell(i) = 1 \\ (\sigma(h(i)), \sigma(b(i))) & \text{otherwise} \end{cases} \tag{4}$$

For two labels i and j, and a substitution σ, if σ is a unifier of i and j then we shall say that i, j are σ-*unifiable*. We shall use $(i, j)\sigma$ to denote both that t and s are σ-unifiable and the result of their unification. In particular

$$\forall i, j, k \in \Im, (i, j)\sigma = k \text{ iff } \exists\sigma : \sigma(i) = \sigma(j) \text{ and } \sigma(i) = k$$

On this basis we may define several specialised, logic-dependent notions of σ-unification characterizing the various modal logic. The first step in order to define the unifications characterizing the various modal logic is to define unifications (axiom unifications) corresponding to the modal axioms. Then in the same way a modal logic is obtained by combining several axiom we define combined unifications, that, when applied recursively produce the logic unifications.

The general form of a σ^A unification is:

$$(i,j)\sigma^A \iff (f_A(i), g_A(j))\sigma \text{ and } C^A$$

where f_A and g_A are given logic-dependent functions from labels to labels and C^A is a set of constraints (see Governatori (1997); Artosi et al. (1998); Gabbay and Governatori (1998a) for example of logic unifications).

A combined unification $\sigma^{A_1\cdots A_n}$ is generally defined as the combination of the axiom unifications for the axioms characterizing the logic

$$(i,j)\sigma^{A_1\cdots A_n} \iff \begin{cases} (i,j)\sigma^{A_1} & C^{A_1} \\ \vdots & \vdots \\ (i,j)\sigma^{A_n} & C^{A_n} \end{cases}$$

Applying recursively the above $\sigma^{A_1\cdots A_n}$ unification we obtain the logic unification σ_L.

$$(i,j)\sigma_L = \begin{cases} (i,j)\sigma^{A_1\cdots A_n} \\ (c^n(i), c^m(j))\sigma^{A_1\cdots A_n} \end{cases}$$

where $w_0 = (s^n(i), s^m(j))\sigma_L$.

As is usual the meaning of an unification is that the denotation of the terms have non-null intersection. However, in some cases, the information encoded in the labels are not enough to determine whether two labels unifies, and we need information from other labels. For example let us assume a non serial modal logic and the labels $i = (W_1, w_0)$, $j = (W_2, w_0)$, and $k = (w_1, w_0)$. According to the meaning of the labels, both i and j denote the set of world accessible from the actual world w_0, while k denotes a world accessible from it. Since our logic is not serial the set of world accessible from w_0 may be empty; however, this is not the case since the non-emptiness of such a set is granted by k. This is the reason for the next unification.

Definition 5. Let $\mathcal{L}$ be a set of labels. Then $(i,j)\sigma_L^{\mathcal{L}}$ iff

1. $(i,j)\sigma_L$ or
2. $\exists k \in \mathcal{L}, \exists n, m \in \mathbb{N}$ such that
   - $(s^n(i), k)\sigma_L^{\mathcal{L}} = (s^m(j), k)\sigma_L^{\mathcal{L}}$ and
   - $(c^n(i), c^m(j))\sigma_L^{\mathcal{L}}$ where $w_0 = (s^n(i), k)\sigma_L^{\mathcal{L}}$

Traditionally formulas in sequents are evaluated as true if they occur in the antecedent, otherwise as false. When we move formulas from one side to the other we have to change their signs, but their contents are left unchanged. Since we use labelled formulas we have to move formulas as well as their labels. In section 2.3 we defined two translation functions from labels to modalities: each one is the opposite of the other. As we shall see, the first translation function is applied to labels occurring in the antecedent and the latter for labels in the consequent.

So when a label moves from the antecedent to the consequent (or the other way around) it changes its sign; where the sign of a label is defined as follows:

DEFINITION 6. For any label i the *specular image* of i, denoted by $\bar{\imath}$ is defined as follows:

$$\bar{\imath} = \begin{cases} i & i \in \Phi_A \\ \bar{\imath} & i \in \Phi_C \cup \Phi_V \\ (\overline{h(i)}, \overline{b(i)}) & \text{otherwise} \end{cases}$$

Furthermore the specular image of a label i satisfies the following properties:

1. If i is a label so is $\bar{\imath}$, similarly for $i \in \Phi_C$ and $i \in \Phi_V$;
2. $\bar{\bar{\imath}} = i$;
3. $\phi^+(\bar{\imath}) = \phi^-(i)$,
   $\phi^-(\bar{\imath}) = \phi^+(i)$;

We further assume that a constant and its specular immage unify, and the result of their unification is the specular immage if the result occurs in the consequent of a sequent, otherwise it is the label itself.

For specular images we can prove

LEMMA 7.

1. *For all* $i, j \in \Im$, $(\bar{\imath}, j)\sigma_L = \overline{(i, \bar{\jmath})\sigma_L}$;
2. *for all* $i, j \in \Im$, $(i, j)\sigma_L^{\mathcal{L}}$ *iff* $(\bar{\imath}, j)\sigma_L^{\mathcal{L}}$.

Proof. By case inspection. □

4 Modal Sequents

A draw back of standard modal sequents is that they define consequence relations for set of modal formulas, but they do not provide a true notion of modal consequence. Moreover it is defined for an (the) actual world. Labels give a first partial relief to this problem insofar as they define a modal consequence with respect to an (the) actual world. However this solution is not general enough. Semantically we can jump from a world to another an set the latter as the current actual world, establish a modal consequence relation with respect to the world, using it to draw inferences, and then we can carry the information thus obtained to another world or back to the original actual world. Composing and decomposing labels corresponds to this mechanism, and KEM labels are very well-suited to this task (see Gabbay and Governatori (1998a,b)). However, this is just the first step in order to define a general modal consequence relation, what we need is to introduce connectives/operators wherever in the formula, not only as main ones. In the next section we show how to achieve this result.

4.1 Inference Rules

The heart of $\mathscr{LMS}$ is constituted by the following sequent rules which are designed to work both as inference rules (to make deductions from both the declarative and the labelled part of wff formulas), and as ways of manipulating labels during proofs.

Axiom

$$A \vdash A$$

Negation

$$\frac{\Gamma, A : i \vdash \Delta}{\Gamma \vdash \neg A : \bar{\imath}, \Delta} \vdash \neg \qquad \frac{\Gamma \vdash A : i, \Delta}{\Gamma, \neg A : \bar{\imath} \vdash \Delta} \neg \vdash$$

Conjunction

$$\frac{\Gamma, A : i, B : j \vdash \Delta}{\Gamma, (A : c^n(i) \wedge B : c^n(j)) : s^n(i) \vdash \Delta} \wedge \vdash$$

where $s^n(i)$ is a segment shared by i and j.

$$\frac{\Gamma \vdash A : i, \Delta \qquad \Gamma' \vdash B : j, \Gamma'}{\Gamma, \Gamma' \vdash (A : c^n(i) \wedge B : c^m(j)) : (s^n(i), s^m(j))\sigma_L^{\mathcal{L}}, \Delta, \Delta'} \vdash \wedge$$

Cut

$$\frac{\Gamma \vdash A : i, \Delta \qquad \Gamma', B : j \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

if $(s^n(i), s^m(j))\sigma_L^{\mathcal{L}}$ and $\phi^-(c^n(i))A = \phi^+(c^m(j))B$. Moreover all the constants occurring in $c^n(i)$ and $c^m(j)$ do not occur in $\Gamma, \Gamma', \Delta, \Delta'$.

Contraction

$$\frac{\Gamma, A : i, B : j \vdash \Delta}{\Gamma, A : i \vdash \Delta} \qquad \frac{\Gamma \vdash A : i, B : j, \Delta}{\Gamma \vdash A : i, \Delta}$$

if $\ell(i) > \ell(j)$ and $\exists n$ such that $\phi(c^n(i))A = \phi(c^n(j))B$ and $s^n(i) = s^n(j)$.

Weakening

$$\frac{\Gamma \vdash A : i, \Delta}{\Gamma \vdash A : i, B : j, \Delta} \qquad \frac{\Gamma, A : i \vdash \Delta}{\Gamma, A : i, B : j \vdash \Delta}$$

Given a set of lwff Γ we shall use $\Gamma^{\mathcal{L}}$ to denote the set of labels occurring in Γ.

Modal rule

$$\frac{\Gamma, A_1 : j_1, \ldots, A_n : j_n \vdash B : j_0, \Delta}{\Gamma, A_1 : j_1 : i_1, \ldots, A_n : j_n : i_n \vdash B : j_0 : k_0, \Delta} RM$$

if

- $j_0 : k_0 = (\bar{p}, \bar{q})\sigma_L^{\mathcal{L}}$, $0 \leq p, q \leq n$; or
- $\forall \bar{p}, \bar{q}((\bar{p}, j_0 : k_0)\sigma_L^{\mathcal{L}}, (\bar{q}, j_0 : k_0)\sigma_L^{\mathcal{L}})\sigma_L^{\mathcal{L}}$

where $\mathcal{L} = \bigcup_{1 \leq p \leq n} j_p : i_p \cup \Gamma^{\mathcal{L}} \cup \Delta^{\mathcal{L}}$ and $\bar{p} = j_p : i_p$ or $\bar{p} = j_0 : k_0$.

4.2 Derived Rules

Introduction of disjunction and implication The rules

$$\frac{\Gamma, A : i \vdash \Delta \qquad \Gamma', B : j \vdash \Delta'}{\Gamma, \Gamma', (A : c^n(i) \vee B : c^m(j)) : (s^n(i), s^m(j))\sigma_L^{\mathcal{L}} \vdash \Delta, \Delta'} \vdash \vee$$

$$\frac{\Gamma \vdash A : i, B : j, \Delta}{\Gamma \vdash (A : c^n(i) \vee B : c^n(j)) : s^n(i), \Delta} \vee \vdash$$

where $s^n(i)$ is a segment shared by i and j; and

$$\frac{\Gamma, A : i \vdash B : j, \Delta}{\Gamma \vdash (A : c^n(\bar{\imath}) \rightarrow B : c^n(j)) : s^n(j), \Delta} \vdash\rightarrow$$

where $s^n(j)$ is a segment shared by i and j

$$\frac{\Gamma \vdash A : i, \Delta \qquad \Gamma', B : j \vdash \Delta'}{\Gamma, \Gamma', (A : c^n(\bar{\imath}) \rightarrow B : c^m(j)) : (s^n(\bar{\imath}), s^m(j))\sigma_L^{\mathcal{L}} \vdash \Delta, \Delta'} \rightarrow\vdash$$

are derived rules. Here we prove only $\vee \vdash$, the others are proofs are similar.

$$\frac{\dfrac{\dfrac{\dfrac{\Gamma, A : i \vdash \Delta}{\Gamma \vdash \neg A : \bar{\imath}, \Delta} \vdash \neg \qquad \dfrac{\Gamma', B : j \vdash \Delta'}{\Gamma' \vdash \neg B : \bar{\jmath}, \Delta'} \vdash \neg}{\Gamma, \Gamma' \vdash (\neg A : c^n(\bar{\imath}) \wedge \neg B : c^m(\bar{\jmath})) : (s^n(\bar{\imath}), s^m(\bar{\jmath}))\sigma_L^{\mathcal{L}}, \Delta, \Delta'} \vdash \wedge}{\Gamma, \Gamma', \neg(\neg A : c^n(\bar{\bar{\imath}}) \wedge \neg B : c^m(\bar{\bar{\jmath}})) : \overline{(s^n(\bar{\imath}), s^m(\bar{\jmath}))\sigma_L^{\mathcal{L}}} \vdash \Delta, \Delta'} \neg \vdash}{\Gamma, \Gamma', (A : c^n(i) \vee B : c^m(j)) : (s^n(i), s^m(j))\sigma_L^{\mathcal{L}} \vdash \Delta, \Delta'} \; RM \text{ and cut}$$

Another bunch of derived rule is the "semantic" version of the α-rules. For example

$$\frac{\Gamma, A : i, B : j \vdash \Delta}{\Gamma, (A : c^n(i) \wedge B : c^m(j)) : (s^n(i), s^m(j))\sigma_L^{\mathcal{L}} \vdash \Delta} \wedge_{\sigma_L^{\mathcal{L}}} \vdash$$

that can be derived by using cut and modal rule. Similarly for the other α-rules.

Modus Ponens The modus ponens

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B}$$

is just an instance of the generalized modal version

$$\frac{\Gamma, \vdash (A : i \rightarrow B : j) : k, \Delta \qquad \Gamma' \vdash C : l, \Delta'}{\Gamma, \Gamma' \vdash B : j : c^n(k) : (s^n(k), s^m(l))\sigma_L^{\mathcal{L}}, \Delta, \Delta'} \; MP$$

where $\phi(i : c^n(k))A = \phi(c^m(l))C$ e $(s^n(k), s^m(l))\sigma_L^{\mathcal{L}}$

On the contrary modus tollens can be derived without limitation only in its propositional version, whereas modal version requires some complex conditions. This is due to the directionality of modalities and negation.

Modus tollens

$$\frac{\vdash (A : i \rightarrow B : j) : k, \quad \vdash C : l}{\vdash \neg A : i : \bar{k}}$$

if $\exists \bar{k} \in \Im, \exists m, p \in \mathbb{N}$ such that $(k, \bar{k})\sigma_L^{\mathcal{L}} = k'$ and $(s^m(j : \bar{k}), s^p(l))\sigma_L^{\mathcal{L}}$, where $\phi(c^m(j : \bar{k}))B$, and $\phi(c^p(l))C$ are complementary.

Introduction of modalities The rules for introducing modalities

$$\frac{\Gamma, A : i \vdash \Delta}{\Gamma, \phi^+(c^n(i))A : s^n(i) \vdash \Delta}\phi\vdash \qquad \frac{\Gamma \vdash A : i, \Delta}{\Gamma \vdash \phi^-(c^n(i))A : s^n(i), \Delta}\vdash\phi$$

where the constants occurring in $\phi(c^n(i))$ do not occur elsewhere, are derived rules.

$$\frac{\dfrac{(1)\phi^-(c^n(i))A \vdash \phi^-(c^n(i))A}{(2)\phi^-(c^n(i))A : j \vdash \phi^-(c^n(i))A : s^n(i)} \quad (3)\Gamma \vdash A : i, \Delta}{(4)\Gamma \vdash \phi^-(c^n(i))A : s^n(i), \Delta}$$

The relevant step is step 2, which has obtained from 1 by an application of the modal rule. Notice that we introduce on the antecedent a label j that unifies with $s^n(i)$. At this point we can apply the cut rule to obtain the desired result.

Elimination of modalities The rules for eliminating the modalities

$$\frac{\Gamma \vdash \mathfrak{m}A : i}{\Gamma \vdash (\mathfrak{n}A : j) : k : i} \qquad \frac{\mathfrak{m}A : i \vdash \Gamma}{(\mathfrak{n}A : j) : k : i \vdash \Gamma}$$

where $\mathfrak{m}A = \phi(k)\mathfrak{n}\phi(j)A$ and the constants occurring in j and k do not occur elsewhere, are derived rules;

$$\frac{\Gamma \vdash \mathfrak{m}A : i \quad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A : j^* \vdash A : j}\text{Modal rule}}{\mathfrak{n}A : j^* \vdash \mathfrak{n}A : j}\text{Introduction of }\mathfrak{n}}{(\mathfrak{n}A : j^*) : k^* \vdash (\mathfrak{n}A : j) : k}\text{Modal rule}}{(\mathfrak{n}A : j^*) : k^* : i^* \vdash (\mathfrak{n}A : j) : k : i}\text{Modal rule}}{\Gamma \vdash (\mathfrak{n}A : j) : k : i}\text{cut}$$

It is worth noting that these rules allow a general treatment of modalities. In particular, given a formula such as $\Diamond\Box\Diamond A : i$ in the consequent of a sequent, we are able to translate each modality even if it is not the main (most external) operator of the formula. Suppose we want to translate just $\Box$. In this case the elimination of this operator produces the following results

$$\Diamond((\Diamond A) : w_2, w_0) : i$$

Necessitation The necessitation rule

$$\frac{\vdash A}{\vdash \Box A}$$

is also a derived rule. In fact it can be derived as follows:

$$\frac{\dfrac{\vdash A : i}{\vdash A : i : (w_1, w_0)}\text{modal rule}}{\vdash \Box(A : i)}\vdash\Box$$

Notice that we have applied the modal rule only with respect to the most external label (w_1, w_0), for which $((w_1, w_0), (w_1, w_0))\sigma_L^{\mathcal{L}} = (w_1, w_0)$ holds; namely it satisfies the first condition for the applicability of the modal rule. We show now that the application of the modal rule only with respect to the most external label is safe.

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash A : i}{\vdash \phi^-(i)A} \vdash \phi^-(i)}{\vdash \phi^-(i)A : (w_1, w_0)} \text{Modal Rule}}{\vdash \Box\phi^-(i)A} \vdash \Box}{\vdash \Box(A : i)} \text{elimination of } \phi^-(i)$$

5 Soundness and Completeness

Basically, we have to show that (1) the rules and the axioms corresponding to a given Hilbert system L for modal logic are respectively derived rules and theorems in $\mathscr{LMS}$, and (2) the rules of $\mathscr{LMS}$ are sound with respect to the semantic conditions for L. In what follows we assume that the Hilbert system L is complete with respect to the appropriate Kripke models.

THEOREM 8. *If* $\vdash_L A$ *then* $\vdash_{\mathscr{LMS}} A$.

Proof. In Section 4.2 we have already seen how to prove modus ponens and necessitation. Modal axioms are derivable as follows:

$$\cfrac{\cfrac{\cfrac{A \vdash A}{A : i \vdash A : j}\text{Modal rule}}{\vdash A : \bar{\imath} \to A : j} \vdash\to}{\vdash \phi^-(i)A \to \phi^-(j)A} \vdash \phi$$

where $(i, j)\sigma^{\mathcal{L}}_L$. This proof relies on the fact that each σ^A-unification corresponds to a generalization including necessitation and self recursion of the modal axiom A, and the various $\sigma^{\mathcal{L}}_L$ are built upon the σ^A of the axioms characterizing the logic L, see Governatori (1997); Artosi et al. (1998). □

Let us first define some functions which map labels into elements of Kripke models. Given a model $\mathcal{M} = \langle W, R, v\rangle$, such functions translate labels into elements of $\mathcal{M}$ according to the structure of the labels.

Let g be a function from $\Im$ to $\wp(\mathcal{W})$ thus defined:

$$g(i) = \begin{cases} h(i) = \{h(i)\} & \text{if } h(i) \in \Phi_C \\ h(i) = \{w_i \in \mathcal{W} : g(b(i))Rw_i\} & \text{if } h(i) \in \Phi_V \end{cases}$$

The above function is not defined for composed labels, i.e., labels of the form $i : j$. However it can be extended to them by stipulating that $g(s^1(i)) = g(h(j))$.

Let r be a function from $\Im$ to R thus defined:

$$r(i) = \begin{cases} \emptyset & \text{if } l(i) = 1 \\ g(i^1)Rg(i^2), \ldots, g(i^{n-1})Rg(h(i)) & \text{if } l(i) = n > 1 \end{cases}$$

Let f be a function from LS-formulas to v thus defined:

$$f(SA, i) =_{def} v(A, w_j) = S$$

for all $w_j \in g(i)$.

As second step, we need the following lemma.

Lemma 9. *For any* $i, k \in \Im$ *if* $(i,k)\sigma_L$ *then* $g(i) \cap g(k) \neq \emptyset$.

Proof. See Artosi et al. (1998); Governatori (1997) □

This lemma shows that if two labels unify, then the result of their σ^L-unification corresponds to an element of the appropriate model. In this way, we are able to build the Kripke model for the labels involved in a $\mathscr{LMS}$ proof, and so we can check every rule of $\mathscr{LMS}$ in a standard semantic setting:

Theorem 10. $\vdash_{\mathscr{LMS}(L)} A \Rightarrow \models_L A$.

Theorem 11. $\models_L A \iff \vdash_L A$.

From theorems 8, 10, and 11 we obtain:

Theorem 12. $\vdash_{\mathscr{LMS}} A \iff \models_L A$.

6 Conclusions

In this paper we have shown that introducing KEM label formalism in a classical sequent calculus, the resulting system we obtain enjoys some interesting properties: it is uniform, in the sense that the deductive framework remains constant among different modal logics; it seems to be flexible enough to deal with other logics; finally, in spite of the logical machinery used in defining the system, we claim its naturalness, in so far as the idea behind is very simple and easy to grasp.

Recently, several proposals have been put forth to find a general framework for modal sequents. We refer, in particular, to hypersequents (Avron, 1996), multidimensional sequents (Došen, 1985; Cerrato, 1996; Masini, 1992), and other versions of labelled sequent calculi (Goré, 1995; Mints, 1997). Nevertheless, all these approaches define the notion of modal derivability in terms of a relativized classical consequence relation and are mainly concerned with the eliminability of cut. In this paper we have taken a different view of what a suitable framework looks like. Accordingly, the main interest in the sequent system just presented is that it provides a general definition of modal consequence relation. Roughly speaking, this means that we can draw inferences with respect to a given world w_i and then move to another world w_j where new inferences can be drawn taking into account the semantical conditions corresponding to the previous inferential steps. An immediate consequence of this approach is that we must be able to manipulate modal operators wherever in a given formula. Basically, this fact has involved a new and more general definition of distributivity of modal operators with respect to boolean ones. On the other hand, keeping trace in the label language of the world path involved in the proof we are not forced to change the basic structure of the sequents for the propositional case.

A final point can be remarked as a matter for future works. It is well-known that for every tableaux proof for a formula A it is possible to build a corresponding (reverse) sequent proof for it. The label formalism we have presented was originally designed for a tableau-like system for modal logics called KEM (see Artosi et al. (1998); Governatori (1997)), where the cut can be restricted to an analytic version; moreover KEM can be extended with the modal generalization of the rules we have proposed for $\mathscr{LMS}$, so it is a suitable tools for such a transformation. Finally our proof method enjoys an interesting property: since the order in which modal principles are applied in the proof is stored in the unifications, it is not hard to reconstruct a Hilbert style proof for A from the order of unifications. This is important because in this way we can produce constructive proofs without references to non-constructive (semantic) methods or to external resources such as labels or other devices.

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